Generation of compound non-Gaussian random processes with a given correlation function

S. Primak*
Department of Electrical and Computer Engineering, The University of Western Ontario, London, Ontario, Canada N6A 5B9
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A compound representation of random processes is considered. Each independent component of such a process is considered as the solution of the proper stochastic differential equation (SDE). This guarantees that the process obtained is stationary and ergodic. The analytical expressions are developed for nonlinear coefficients of the generating SDE. Theoretical results are compared with numerical simulation.

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I. INTRODUCTION

A need for accurate computer simulation of random processes with a given non-Gaussian probability density function (PDF) and certain correlation properties arises in applications ranging from communications, mechanical vibrations, and sea surface clutter modeling to large scale astronomical phenomena [1–3]. Conventional methods, based either on linear filtering or memoryless nonlinear transformation of Gaussian correlated processes are limited as far as analytical results are concerned [4,5]. Moreover, an accurate and simultaneous reproduction of the correlation function (CF) and PDF would require complicated optimization techniques. A general approach to generate a stationary random process with a given PDF and an exponential-type CF based on nonlinear transformation, with memory of the white Gaussian noise (WGN) was presented in Refs. [6,7]. This approach is based on the treatment of the process with the required characteristics as the solution of a proper stochastic differential equation (SDE) [8]. Such an interpretation is attractive because it takes advantage of the Markov process theory [9]. As shown in Refs. [7,10] it is possible to generate a continuous random process with any given PDF and exact exponential correlation function.

In this paper we develop a procedure allowing us to model non-Gaussian random processes with an arbitrary correlation function and marginal PDF restricted to a specific class of so-called compound PDF. In contrast to the approach presented in Refs. [11–13], our method generates an ergodic stationary Markov process.

II. SDE MODEL OF THE EXPONENTIALLY CORRELATED RANDOM PROCESS

In order to make this paper self-explanatory, we reproduce some basic equations, earlier obtained in Ref. [7], which allow us to generate an exponentially correlated random process with an arbitrary probability density function (PDF). The solution of a SDE (Ito form [9]),

$$\dot{x}=f(x)+g(x)\xi(t), \tag{1}$$

is a Markov random process, whose PDF $p(x,t)$ [and the transition probability density function $\pi(x,t|x_0,t_0)$] obeys the Fokker-Planck equation (FPE) [9]

$$-\frac{\partial}{\partial t}p(x,t)=\frac{\partial}{\partial x}[K_1(x)p(x,t)]-\frac{1}{2}\frac{\partial^2}{\partial x^2}[K_2(x)p(x,t)], \tag{2}$$

where $\xi(t)$ is a WGN of unit intensity, and

$$K_1(x)=f(x), \tag{3}$$

$$K_2(x)=g^2(x) \tag{4}$$

are the drift and diffusion of the Markov process $x(t)$. The nonstationary PDF $p(x,t)$ of the process $x(t)$, which converges to the stationary PDF $p_x(x)$ when t approaches infinity, i.e.,

$$\lim_{t\to\infty}p(x,t)=p_x(x).$$

There is a simple relation between $K_1(x)$, $K_2(x)$, and $p_x(x)$ [9]:

$$p_x(x)=\frac{C}{K_2(x)}\exp\left[2\int_a^x\frac{K_1(x)}{K_2(x)}dx\right], \tag{5}$$

where a constant C is chosen to normalize the PDF $p_x(x)$.

At the same time, the correlation function $K_x(\tau)=\langle x(t)x(t+\tau)\rangle$ can be considered as the solution of the following ordinary differential equation [9]:

$$\frac{d}{d\tau}K_x(\tau)=\langle x(t)K_1(x(t+\tau))\rangle \tag{6}$$

with the initial condition

$$K_x(0)=\sigma_x^2=\langle(x-\langle x\rangle)^2\rangle=\langle(x-m_x)^2\rangle. \tag{7}$$

Here $\langle\bullet\rangle$ stands for the statistical average over the realizations [9]. If one chooses

$$K_1(x)=-\lambda(x-m_x), \tag{8}$$

then Eq. (6) has the solution of the form

$$K_{xx}(\tau)=\sigma_x^2\exp(-\lambda|\tau|). \tag{9}$$

*Electronic address: primak@engga.uwo.ca

After substituting Eq. (8) into Eq. (5) and solving for $K_2(x)$, one can obtain that

$$K_2(x) = -\frac{2\lambda}{p_x(x)}\int_{-\infty}^{x}(x-m_x)p_x(x)dx. \tag{10}$$

The drift $K_1(x)$ and the diffusion $K_2(x)$ now define the SDE:

$$\dot{x} = -\lambda(x-m_x) + \sqrt{-\frac{2\lambda}{p_x(x)}\int_{-\infty}^{x}(x-m_x)p_x(x)dx}\,\xi(t), \tag{11}$$

whose solution has the given stationary PDF $p_x(x)$ and exponential correlation function (9). In turn, SDE (11) can be numerically simulated, using a technique suggested in Ref. [8], providing one with a convenient tool of generating non-Gaussian exponentially correlated random processes.

III. SDE REPRESENTATION OF COMPOUND PROCESSES

Following Refs. [11–13], let us consider the random process $x(t)$ as being a product of a zero mean Gaussian random process $n(t)$ with the marginal PDF $p_n(n)$, and (t) a symmetric exponentially correlated random process $s(t)$, with the marginal PDF $p_s(|s|)$, which is independent of $n(t)$:

$$x(t) = n(t)s(t). \tag{12}$$

We will refer to the process $x(t)$ as a compound process to emphasize that it is obtained as a product of two processes. The random processes $n(t)$ and $s(t)$ can be considered as the components of the corresponding compound process $x(t)$.

Let

$$B_n(\tau) = K_n(\tau) = \langle n(t)n(t+\tau)\rangle \tag{13}$$

and

$$B_s(\tau) = K_s(\tau) = \langle s(t)s(t+\tau)\rangle \tag{14}$$

be the correlation (covariation) functions of the processes $n(t)$ and $s(t)$, respectively [both $n(t)$ and $s(t)$ have zero mean]. Then, using the independence of $n(t)$ and $s(t)$, one can write

$$\begin{aligned} B_x(\tau) &= \langle x(t)x(t+\tau)\rangle = \langle n(t)n(t+\tau)s(t)s(t+\tau)\rangle \\ &= \langle n(t)n(t+\tau)\rangle\langle s(t)s(t+\tau)\rangle = B_n(\tau)B_s(\tau). \end{aligned} \tag{15}$$

The last equality means that the correlation function $R_{xx}(\tau)$ of the compound process $x(t)$ is just a product of the correlation functions $R_n(\tau)$ and $R_s(\tau)$ of its components $n(t)$ and $s(t)$.

The marginal PDF of the Gaussian component $n(t)$ can be written as

$$p_n(n) = \frac{1}{\sqrt{2\pi B_n(0)}}\exp\left[-\frac{n^2}{2B_n(0)}\right] \tag{16}$$

and, taking into account the independence of its components $n(t)$ and $s(t)$, the joint PDF $p_{n,s}(n,s)$ of $n(t)$ and $s(t)$ is

$$p_{n,s}(n,s) = \frac{1}{\sqrt{2\pi B_n(0)}}\exp\left[-\frac{n^2}{2B_n(0)}\right]p_s(s). \tag{17}$$

This immediately produces

$$\begin{aligned} p_x(x) &= 2\int_0^{\infty}\frac{1}{\sqrt{2\pi B_n(0)}}\exp\left[-\frac{x^2}{2B_n(0)s^2}\right]\frac{p_s(s)}{s}ds \\ &= \int_0^{\infty}\frac{1}{\sqrt{2\pi B_n(0)}}\exp\left[-\frac{x^2}{2B_n(0)s^2}\right]\frac{\hat{p}_s(s)}{s}ds, \end{aligned} \tag{18}$$

where

$$\hat{p}_s(s) = \begin{cases} 0 & \text{if } s<0 \\ 2p_s(s) & \text{if } s\geqslant 0. \end{cases} \tag{19}$$

As we can see, the marginal distribution of $x(t)$ does not depend on correlation properties of its components. Thus, if it is possible to find a distribution $p_s(s)$ that gives rise to the desired marginal distribution $p_x(x)$, then one can adjust the correlation properties of components to achieve the desired correlation properties of the compound process itself. Therefore, the marginal distribution of component $s(t)$ allows us to obtain the desired distribution, while the desired correlation properties are defined by the correlation properties of the Gaussian component.

To prove the last proposition, let us assume that the desired correlation function of the process $x(t)$ should be a sum of exponents (we assume here that any desired correlation function can be well approximated by a rational correlation function)

$$K_x(\tau) = \sum_{k=1}^{K}\alpha_k\exp[s_k\tau], \tag{20}$$

where K is the order of approximation. Poles s_k and the corresponding residue α_k should appear in complex-conjugate pairs. Let $\bar{\alpha}$ be defined as

$$\bar{\alpha} = \max\{\mathrm{real}(s_k)\} < 0, \tag{21}$$

which means that $K_x(\tau)\to 0$ when $\tau\to\infty$. If we then choose

$$K_s(\tau) = K_s(0)\exp\left[-\frac{\bar{\alpha}}{2}\tau\right] \tag{22}$$

and

$$K_s(\tau) = \frac{K_x(\tau)}{K_s(0)}\exp\left[\frac{\bar{\alpha}}{2}\tau\right] = \sum_{k=1}^{K}\frac{\alpha_k}{K_s(0)}\exp\left[\left(s_k+\frac{\bar{\alpha}}{2}\right)\tau\right], \tag{23}$$

then

$$\max\left\{\mathrm{real}\left(s_k+\frac{\bar{\alpha}}{2}\right)\right\} = -\frac{\bar{\alpha}}{2} < 0, \tag{24}$$

so $K_n(\tau)\to 0$ when $\tau\to\infty$ and

$$K_n(\tau)K_s(\tau) = K_x(\tau). \tag{25}$$

If we can then find a function $p_s(s)$ that gives rise to the random process $x(t)$ with the given PDF $p_x(x)$, then an SDE generating a random process with a given marginal PDF and given correlation function is found. The class of PDF, allowing such a representation, can be found in Refs. [11–13]. The process $s(t)$ can be obtained using SDE (11). The Gaussian process, with the correlation function $K_n(\tau)$, can be generated as a component of the solution $\mathbf{n}(t)$ of K-dimensional SDE of the form [14]:

$$\frac{d}{dt}\mathbf{n}(t)=\mathbf{A}\mathbf{n}(t)+\mathbf{B}\Xi(t), \tag{26}$$

where $\mathbf{A}$, $\mathbf{B}$ are the constant matrices, and $\Xi(t)$ is the K-dimensional WGN with independent components.

IV. AN EXAMPLE OF A NARROW BAND *K*-DISTRIBUTED RANDOM PROCESS

Using the random process with a marginal distribution,

$$p_x(x)=K_0(ax), \tag{27}$$

and the covariation function in the following form:

$$K_{xx}(x)=\sigma^2\exp(-\lambda_0|\tau|)\cos(\omega\tau),\quad \sigma^2=(2\pi)/a^2, \tag{28}$$

the desired process $x(t)$ can be represented as a product of a normally distributed random process $x_n(t)$, with covariation function

$$K_{x_nx_n}(x)=\sigma^2\exp\left(-\frac{\lambda_0}{2}|\tau|\right)\cos(\omega\tau), \tag{29}$$

and the m-distributed random Λ process $s(t)$ with $m=0.5$ and $\Omega=1$ [7].

In this case, the process $x_n(t)$ is generated by the SDE

$$\ddot{x}_n+\frac{\lambda_0}{\omega}\dot{x}_n+\omega^2x_n=\sigma\xi_n(t), \tag{30}$$

and $s(t)$ is generated by [7]

$$\dot{s}=-\frac{\lambda_0}{2}\left(s-\sqrt{\frac{2}{\pi}}\right)+F(s)\xi(t), \tag{31}$$

where $F(s)$, according to Eq. (10), is given by

$$F(s)=\sqrt{-\frac{2\lambda}{K_0(as)}\int_{-\infty}^{s}(x-m_x)K_0(ax)dx}$$
$$=\sqrt{\frac{\lambda_0\sqrt{2}\Gamma\left(\frac{1}{2},0,\frac{s^2}{2}\right)}{\pi}}, \tag{32}$$

and $\Gamma(\bullet,\bullet,\bullet)$ is the generalized γ function [15]. The results of the numerical simulation for $a=1$ and $\gamma=10^{-9}$ s are given in Figs. 1 and 2.

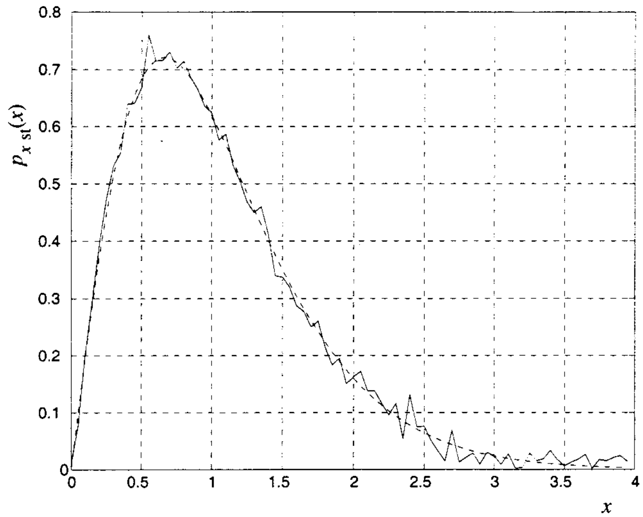

FIG. 1. Theoretical (dashed line) and estimated (solid line) PDF, obtained from the solution of Eqs. (30) and (31).

V. CONCLUSIONS

This paper addresses the problem of modeling a compound non-Gaussian random process with a given PDF and correlation function. It was suggested that the desired random process is represented by a product of the correlated Gaussian random process and the exponentially correlated non-Gaussian random process, which are independent of each other. Both the components of the compound process are represented as solutions of corresponding SDE. These methods enable one to generate an ergodic random process, in sharp contrast to the method frequently used in the literature [11–13]. Both components can be completely described by their corresponding transition probability density functions, and thus, using the Markov property of SDE solution, the joint probability density function of any order can be obtained. Part of these results can be found in Ref. [7], and are the subject of a future publication. Numerical simulation confirms the theoretical derivations. We believe that the sug-

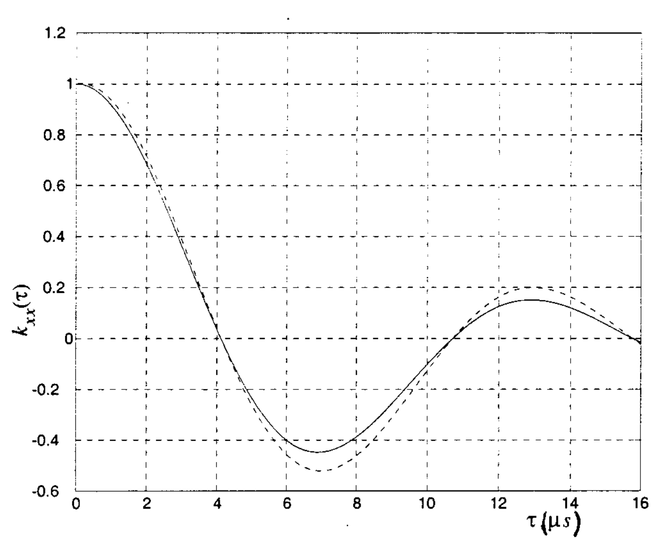

FIG. 2. Theoretical (dashed line) and estimated (solid line) CF, obtained from the solution of Eqs. (30) and (31).

gested approach can be extended to the case when the desired distribution has infinite moments—one may try to use nonstationary Gaussian components of the compound process with growing variance. However, this is a matter for further research.

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